

SHARP ERROR BOUNDS FOR JACOBI EXPANSIONS AND GENGENBAUER-GAUSS QUADRATURE OF ANALYTIC FUNCTIONS

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ABSTRACT. This paper provides a rigorous and delicate analysis for exponential decay of Jacobi polynomial expansions of analytic functions associated with the Bernstein ellipse. Using an argument that can recover the best estimate for the Chebyshev expansion, we derive various new and sharp bounds of the expansion coefficients, which are featured with explicit dependence of all related parameters and valid for degree $n \geq 1$. We demonstrate the sharpness of the estimates by comparing with existing ones, in particular, the very recent results in [38, SIAM J. Numer. Anal., 2012]. We also extend this argument to estimate the Gegenbauer-Gauss quadrature remainder of analytic functions, which leads to some new tight bounds for quadrature errors.

1. INTRODUCTION

The spectral method employs global orthogonal polynomials or Fourier complex exponentials as basis functions, so it enjoys high-order accuracy (with only a few basis functions), if the underlying function is smooth (and periodic in the Fourier case). The convergence rate $O(n^{-r})$, where n is the number of basis functions involved in a spectral expansion and r is related to the Sobolev-regularity of the underlying function, is typically documented in various monographs on spectral methods [18, 15, 14, 4, 21, 35, 7, 8, 24, 32]. It is also widely appreciated that if the function under consideration is analytic, the convergence rate is of exponential order $O(q^n)$ (for constant $0 < q < 1$). However, there appears very limited discussions of such error bounds (mostly mentioned, but not proved) in [14, 35, 7]. Indeed, as commented by Hale and Trefethen [23], the general idea of such convergence goes back to Bernstein in early nineties, but such results do not appear in many textbooks or monographs, and there is not much uniformity in the constants in the upper bounds.

An important result in Bernstein [5] (1912) (also see [28]) states that u is analytic on $[-1, 1]$, if and only if

$$\sup_{N \rightarrow \infty} \lim_{N \rightarrow \infty} \sqrt[N]{E_N(u)} = \frac{1}{\rho}, \quad E_N(u) = \inf_{v \in P_N} \|v - u\|_{\infty},$$

where P_N is the polynomial space of degree no more than N , and $\rho > 1$ is the sum of the semi-axes of the maximum ellipse \mathcal{E}_{ρ} with foci ± 1 , known as the *Bernstein ellipse*, on and within which u can be analytically extended to. One immediate implication is that the best

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polynomial approximation in the maximum norm enjoys exponential convergence. A more precise estimate for the Chebyshev expansion can be found in various approximation theory texts (see e.g., [31, Theorem 3.8] and [29, Theorem 5.16]):

$$|\hat{u}_n^C| \leq \frac{2M}{\rho^n}, \quad \forall n \geq 0; \quad \|u - S_N^C u\|_\infty \leq \frac{M}{(\rho - 1)\rho^N}, \quad (1.1)$$

where $M = \max_{z \in \mathcal{E}_\rho} |u(z)|$, $\{\hat{u}_n^C\}$ are Chebyshev expansion coefficients of u , and $S_N^C u$ is the partial sum involving the first $N + 1$ terms. One also refers to [33, 12, 31, 6, 29, 36, 37] and the references therein for verification/description of exponential convergence of Fourier, Chebyshev or Legendre expansions. We remark that Gottlieb and Shu et al [20, 19] studied exponential convergence of Gegenbauer expansions (when the parameter grows linearly with n) in the context of defeating the Gibbs phenomenon.

Here, we particularly highlight that a very recent paper of Xiang [38] provided a simple approach to obtain the bounds for Jacobi expansion coefficients of analytic functions on and within the Bernstein ellipse \mathcal{E}_ρ :

$$|\hat{u}_n^{\alpha, \beta}| \leq \frac{2M}{\rho^{n-1}(\rho - 1)} \sqrt{\frac{\gamma_0^{\alpha, \beta}}{\gamma_n^{\alpha, \beta}}} \quad \text{where} \quad \hat{u}_n^{\alpha, \beta} = \frac{1}{\gamma_n^{\alpha, \beta}} \int_{-1}^1 u(x) J_n^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x) dx. \quad (1.2)$$

Here, $\{J_n^{\alpha, \beta}\}(\alpha, \beta > -1)$ are Jacobi polynomials mutually orthogonal with the weight function $\omega^{\alpha, \beta}(x) = (1 - x)^\alpha (1 + x)^\beta$ and with the normalization factor $\gamma_n^{\alpha, \beta}$ (cf. (2.8)). The key step is to insert the Chebyshev expansion $u(x) = \sum_{j=0}^\infty \hat{u}_j^C T_j(x)$ into the Jacobi expansion coefficients and rewrite

$$\hat{u}_n^{\alpha, \beta} = \frac{1}{\gamma_n^{\alpha, \beta}} \sum_{j=0}^\infty \hat{u}_j^C \int_{-1}^1 T_j(x) J_n^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x) dx,$$

so the bound for the Chebyshev coefficient in (1.1) could be used.

The first purpose of the paper is to take a different approach to derive sharp estimates for general Jacobi expansion of analytic functions. The assertion of sharpness is in the following sense:

- (i) The bound for general Jacobi case is tighter than (1.2) (see Remark 2.3).
- (ii) Refined estimates can be obtained for Gegenbauer expansion ($\alpha = \beta > -1$), Chebyshev-type expansion ($\alpha = k - 1/2, \beta = l - 1/2$ for non-negative integers k, l), and Legendre-type expansion ($\alpha = k, \beta = l$ for non-negative integers k, l). The argument can recover the bounds known to be the sharpest (e.g., the Chebyshev case), and some obtained estimates are new and significantly improve the existing ones (see e.g., Remark 2.5).

A second purpose of this work is to extend the argument to analyze Gegenbauer-Gauss quadrature of analytical functions. Recall that the remainder of Gauss-quadrature with the nodes and weights $\{x_j, \omega_j\}_{j=1}^n$, takes the form (see e.g., [13]):

$$E_n[u] = \int_{-1}^1 u(x) \omega(x) dx - \sum_{j=1}^n u(x_j) \omega_j = \frac{1}{\pi i} \oint_{\mathcal{E}_\rho} \frac{q_n(z)}{p_n(z)} u(z) dz, \quad (1.3)$$

where $\{x_j\}_{j=1}^n$ are the zeros of $p_n(x)$, orthogonal with respect to the weight function $\omega(x)$, and

$$q_n(z) = \frac{1}{2} \int_{-1}^1 \frac{p_n(x) \omega(x)}{z - x} dx. \quad (1.4)$$

The estimate of quadrature errors has attracted much attention (see e.g., [11, 10, 3, 17, 13, 16, 25, 26]). Among these results, intensive discussions have been centered around the Chebyshev case and its family, e.g., Chebyshev of the second kind, but with very limited results even for Legendre-Gauss quadrature (see e.g., [9, 27]). In fact, the analysis heavily relies on the availability of explicit expression of $p_n(z)$ on \mathcal{E}_ρ . Armed with a delicate estimate of $q_n(z)$ (in the first part of the paper) and the explicit formula of Gegenbauer polynomial in [39], we are able to derive sharp bound for the Gegenbauer-Gauss quadrature errors.

We remark that there has been much interest in estimating spectral differentiation errors of analytic functions. Tadmor [34] first attempted to estimate the aliasing errors to verify exponential convergence of Fourier and Chebyshev spectral differentiation with a different assumption on analyticity. The results for analyticity characterized by the Bernstein ellipse include Reddy and Weideman [30] for Chebyshev case, and Xie, Wang and Zhao [39] for Gegenbauer spectral differentiation. It is also interesting to point out that Zhang [40, 41, 42] studied superconvergence of spectral interpolation and differentiation. We stress that the analysis apparatuses and arguments in this pipeline are different from these in this work.

The rest of this paper is organized as follows. In Section 2, we provide sharp bounds for general Jacobi expansions of analytic functions, followed by some refined results for Chebyshev-type and Legendre-type expansions. In Section 3, we extend the argument to analyze Gegenbauer-Gauss quadrature errors. In the final section, we provide results to show the sharpness of the bounds by comparing them with existing ones.

2. SHARP BOUNDS FOR JACOBI EXPANSIONS

We derive in this section sharp bounds for Jacobi expansions of functions analytic on and within the Bernstein ellipse \mathcal{E}_ρ .

2.1. Preliminaries. It is known (see e.g., [12]) that the Bernstein ellipse is transformed from the circle

$$\mathcal{C}_\rho = \{w = \rho e^{i\theta} : \theta \in [0, 2\pi]\}, \quad \rho > 1, \quad (2.1)$$

via the conformal mapping: $z = (w + w^{-1})/2$, namely,

$$\mathcal{E}_\rho := \left\{z \in \mathbb{C} : z = \frac{1}{2}(w + w^{-1}) \text{ with } w = \rho e^{i\theta}, \theta \in [0, 2\pi]\right\}, \quad (2.2)$$

where \mathbb{C} is the set of all complex numbers, and $i = \sqrt{-1}$ is the complex unit. It has the foci at ± 1 , and the major and minor semi-axes are

$$a = \frac{1}{2}(\rho + \rho^{-1}), \quad b = \frac{1}{2}(\rho - \rho^{-1}), \quad (2.3)$$

respectively, so the sum of two semi-axes is ρ . The perimeter of \mathcal{E}_ρ has the bound

$$L(\mathcal{E}_\rho) \leq \pi \sqrt{\rho^2 + \rho^{-2}}, \quad (2.4)$$

which overestimates the perimeter by less than 12 percent (cf. [30]). The distance from \mathcal{E}_ρ to the interval $[-1, 1]$ is

$$d_\rho = \frac{1}{2}(\rho + \rho^{-1}) - 1. \quad (2.5)$$

We see that d_ρ increases with respect to ρ , and $d_\rho \rightarrow 0^+$ as $\rho \rightarrow 1^+$ (so the ellipse reduces to the interval $[-1, 1]$). Thus, by the theory of analytic continuation, we have that *for any*

analytic function u on $[-1, 1]$, there always exists a Bernstein ellipse \mathcal{E}_ρ with $\rho > 1$ such that the continuation of u is analytic on and within \mathcal{E}_ρ . Hereafter, we denote by

$$\mathcal{A}_\rho := \{u : u \text{ is analytic on and within } \mathcal{E}_\rho\}, \quad 1 < \rho < \rho_{\max}, \quad (2.6)$$

where $\mathcal{E}_{\rho_{\max}}$ labels the largest ellipse within which u is analytic. In particular, if $\rho_{\max} = \infty$, u is an entire function.

Throughout this paper, the Jacobi polynomials, denoted by $J_n^{\alpha, \beta}(x)$ (with $\alpha, \beta > -1$ and $x \in I := (-1, 1)$), are normalized as in Szegő [33], i.e.,

$$\int_{-1}^1 J_n^{\alpha, \beta}(x) J_m^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x) dx = \gamma_n^{\alpha, \beta} \delta_{m, n}, \quad (2.7)$$

where $\omega^{\alpha, \beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\delta_{m, n}$ is the Kronecker delta, and

$$\gamma_n^{\alpha, \beta} = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) n! \Gamma(n+\alpha+\beta+1)}. \quad (2.8)$$

In Appendix A, we collect the relevant properties of Jacobi polynomials.

In the analysis, we also use the following property of the Gamma function, derived from [1, Eq. (6.1.38)]:

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} \exp\left(-x + \frac{\theta}{12x}\right), \quad \forall x > 0, \quad 0 < \theta < 1. \quad (2.9)$$

Lemma 2.1. *For any constants a, b , we have that for $n \geq 1$, $n+a > 1$ and $n+b > 1$,*

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \leq \Upsilon_n^{a, b} n^{a-b}, \quad (2.10)$$

where

$$\Upsilon_n^{a, b} = \exp\left(\frac{a-b}{2(n+b-1)} + \frac{1}{12(n+a-1)} + \frac{(a-b)^2}{n}\right). \quad (2.11)$$

Proof. Let θ_1, θ_2 be two constants in $(0, 1)$. We find from (2.9) that

$$\begin{aligned} \frac{\Gamma(n+a)}{\Gamma(n+b)} &= \frac{(n+a-1)^{n+a-1/2}}{(n+b-1)^{n+b-1/2}} \exp\left(-a+b + \frac{\theta_1}{12(n+a-1)} - \frac{\theta_2}{12(n+b-1)}\right) \\ &\leq (n+a-1)^{a-b} \left(1 + \frac{a-b}{n+b-1}\right)^{n+b-1/2} \exp\left(-a+b + \frac{1}{12(n+a-1)}\right) \\ &\leq n^{a-b} \left(1 + \frac{a-b}{n}\right)^{a-b} \exp\left(-a+b + \frac{(a-b)(n+b-1/2)}{n+b-1} + \frac{1}{12(n+a-1)}\right) \\ &\leq n^{a-b} \exp\left(\frac{a-b}{2(n+b-1)} + \frac{1}{12(n+a-1)} + \frac{(a-b)^2}{n}\right) := \Upsilon_n^{a, b} n^{a-b}, \end{aligned}$$

where we used the fact that $1+x \leq e^x$, for real x . \square

Remark 2.1. Applying (2.11) to $\gamma_n^{\alpha, \beta}$ leads to that for $\alpha, \beta > -1$, $n \geq 1$ and $n+\alpha+\beta > 0$,

$$\gamma_n^{\alpha, \beta} \leq \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \Upsilon_n^{\alpha+1, 1} \Upsilon_n^{\beta+1, \alpha+\beta+1}. \quad (2.12)$$

Note that for fixed a and b ,

$$\Upsilon_n^{a, b} = 1 + O(n^{-1}), \quad (2.13)$$

as it behaves like $e^{1/n}$. \square

2.2. Main tools.

Our starting point is the following important representation.

Lemma 2.2. *Let $\{\hat{u}_n^{\alpha,\beta}\}$ be the Jacobi polynomial expansion coefficients given by*

$$\hat{u}_n^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 u(x) J_n^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) dx, \quad \alpha, \beta > -1, \quad n \geq 0. \quad (2.14)$$

If $u \in \mathcal{A}_\rho$ with $\rho > 1$, we have the representation:

$$\hat{u}_n^{\alpha,\beta} = \frac{1}{\pi i} \sum_{j=0}^{\infty} \sigma_{n,j}^{\alpha,\beta} \oint_{\mathcal{E}_\rho} \frac{u(z)}{w^{n+j+1}} dz, \quad n \geq 0, \quad (2.15)$$

where $z = (w + w^{-1})/2$ with $w = \rho e^{i\theta}$, $\theta \in [0, 2\pi]$, and

$$\sigma_{n,j}^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 U_{n+j}(x) J_n^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) dx, \quad n, j \geq 0. \quad (2.16)$$

Here, $U_{n+j}(x)$ is the Chebyshev polynomial of the second kind of degree $n+j$ (cf. (A.5)).

Actually, the formula (2.15)-(2.16) can be obtained by assembling several formulas in Szegő [33], and then using the generating function of $U_k(x)$ (cf. [1]). For the readers' reference, we sketch its derivation in Appendix B.

The establishment of sharp bounds heavily relies on estimating $\sigma_{n,j}^{\alpha,\beta}$. The following explicit formulas follow from (2.16) and some properties of Jacobi polynomials listed in Appendix A. We remark that the formula (2.19) can be found in various books e.g., [12, 29], while the formula (2.20) is due to Heine (see [11]). We also highlight that the formula (2.21) for the general Jacobi case seems new.

Corollary 2.1. *Let $n \geq 0$.*

(i) *For $\alpha = \beta > -1$ (ultraspherical/Gegenbauer polynomial)¹,*

$$\sigma_{n,j}^{\alpha,\alpha} = 0, \quad \text{for odd } j. \quad (2.17)$$

(ii) *For $\alpha = \beta = 1/2$ (Chebyshev polynomial of the second kind),*

$$\sigma_{n,0}^{1/2,1/2} = \frac{\sqrt{\pi}}{2} \frac{(n+1)!}{\Gamma(n+3/2)}; \quad \sigma_{n,j}^{1/2,1/2} = 0, \quad \text{for } j \geq 1. \quad (2.18)$$

(iii) *For $\alpha = \beta = -1/2$ (Chebyshev polynomial),*

$$\sigma_{n,j}^{-1/2,-1/2} = \begin{cases} \frac{2\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+1/2)}, & \text{for even } j, \\ 0, & \text{for odd } j. \end{cases} \quad (2.19)$$

(iv) *For $\alpha = \beta = 0$ (Legendre polynomial),*

$$\sigma_{n,j}^{0,0} = \begin{cases} \frac{2n+1}{2} \frac{\Gamma(l+1/2)}{\Gamma(l+1)} \frac{\Gamma(n+l+1)}{\Gamma(n+l+3/2)}, & \text{for even } j = 2l, \\ 0, & \text{for odd } j. \end{cases} \quad (2.20)$$

¹In this paper, we do not distinguish between ultraspherical and Gegenbauer polynomials.

(v) For general $\alpha, \beta > -1$ (Jacobi polynomial),

$$\begin{aligned} \sigma_{n,j}^{\alpha,\beta} &= \frac{\sqrt{\pi}(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{2\Gamma(n+\alpha+1)} \\ &\times \sum_{m=0}^j \frac{(-1)^m \Gamma(2n+j+m+2)\Gamma(n+m+\alpha+1)}{m!(j-m)!\Gamma(n+m+3/2)\Gamma(2n+m+\alpha+\beta+2)}. \end{aligned} \quad (2.21)$$

Proof. (i). The property (2.17) is a direct consequence of the parity of ultraspherical polynomials.

(ii). For $\alpha = \beta = 1/2$, we find from (A.5) and the orthogonality (2.7)-(2.8) that

$$\begin{aligned} \sigma_{n,j}^{1/2,1/2} &= \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\gamma_{n+j}^{1/2,1/2}}} \frac{1}{\gamma_n^{1/2,1/2}} \int_{-1}^1 J_{n+j}^{1/2,1/2}(x) J_n^{1/2,1/2}(x) (1-x^2)^{1/2} dx \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\gamma_{n+j}^{1/2,1/2}}} \delta_{j,0}, \end{aligned}$$

where $\delta_{j,0}$ is the Kronecker delta. Working out the constant leads to (2.18).

(iii) For $\alpha = \beta = -1/2$, if $j = 2l$, we have

$$\begin{aligned} \sigma_{n,2l}^{-1/2,-1/2} &\stackrel{(A.5)}{=} \frac{1}{\gamma_n^{-1/2,-1/2}} \frac{1}{n+2l+1} \int_{-1}^1 T'_{n+2l+1}(x) J_n^{-1/2,-1/2}(x) (1-x^2)^{-1/2} dx \\ &\stackrel{(A.6b)}{=} \frac{2}{\gamma_n^{-1/2,-1/2}} \int_{-1}^1 T_n(x) J_n^{-1/2,-1/2}(x) (1-x^2)^{-1/2} dx \stackrel{(A.6a)}{=} \frac{2\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+1/2)}, \end{aligned}$$

which, together with (2.17), implies (2.19).

(iv) For $\alpha = \beta = 0$, we derive from [11, Eq. (14)] that

$$\begin{aligned} \sigma_{n,2l}^{0,0} &= \frac{1}{\gamma_{n,0}^{0,0}} \int_{-1}^1 J_n^{0,0}(x) U_{n+2l}(x) dx = \frac{2n+1}{2} \int_0^\pi J_n^{0,0}(\cos \theta) \sin((n+2l+1)\theta) d\theta \\ &= \frac{2n+1}{2} \frac{\Gamma(l+1/2)}{\Gamma(l+1)} \frac{\Gamma(n+l+1)}{\Gamma(n+l+3/2)}, \quad l \geq 0. \end{aligned}$$

This yields (2.20).

(v) The formula (2.21) follows from a combination of (2.8), (A.4) and (A.5). \square

With the aid of Lemma 2.2, we can derive the following estimate, from which our sharp bounds are stemmed.

Lemma 2.3. For any $u \in \mathcal{A}_\rho$ with $\rho > 1$, we have that for $\alpha, \beta > -1$ and $n \geq 0$,

$$|\hat{u}_n^{\alpha,\beta}| \leq \frac{M}{\rho^n} \left(|\sigma_{n,0}^{\alpha,\beta}| + \frac{1}{\rho} |\sigma_{n,1}^{\alpha,\beta}| + \frac{1}{\rho^2} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \frac{1}{\rho^j} \right), \quad (2.22)$$

where $M = \max_{z \in \mathcal{E}_\rho} |u(z)|$ and $\{\sigma_{n,j}^{\alpha,\beta}\}$ are given by (2.16).

Proof. Since $z = (w + w^{-1})/2 \in \mathcal{E}_\rho$ with $w \in \mathcal{C}_\rho$ (cf. (2.1)-(2.2)), we can rewrite $\hat{u}_n^{\alpha,\beta}$ in (2.15) as

$$\begin{aligned}\hat{u}_n^{\alpha,\beta} &= \frac{1}{2\pi i} \sum_{j=0}^{\infty} \sigma_{n,j}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+j+1}} \left(1 - \frac{1}{w^2}\right) dw \\ &= \frac{1}{2\pi i} \sum_{j=0}^{\infty} \sigma_{n,j}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+j+1}} dw - \frac{1}{2\pi i} \sum_{j=0}^{\infty} \sigma_{n,j}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+j+3}} dw \\ &= \frac{1}{2\pi i} \sigma_{n,0}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+1}} dw + \frac{1}{2\pi i} \sigma_{n,1}^{\alpha,\beta} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+2}} dw \\ &\quad + \frac{1}{2\pi i} \sum_{j=0}^{\infty} (\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}) \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+j+3}} dw.\end{aligned}\tag{2.23}$$

Hence, we arrive at

$$\begin{aligned}|\hat{u}_n^{\alpha,\beta}| &\leq \frac{M}{2\pi} \frac{2\pi\rho}{\rho^{n+1}} |\sigma_{n,0}^{\alpha,\beta}| + \frac{M}{2\pi} \frac{2\pi\rho}{\rho^{n+2}} |\sigma_{n,1}^{\alpha,\beta}| + \frac{M}{2\pi} \frac{2\pi\rho}{\rho^{n+3}} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \frac{1}{\rho^j} \\ &= \frac{M}{\rho^n} |\sigma_{n,0}^{\alpha,\beta}| + \frac{M}{\rho^{n+1}} |\sigma_{n,1}^{\alpha,\beta}| + \frac{M}{\rho^{n+2}} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \frac{1}{\rho^j}.\end{aligned}\tag{2.24}$$

This ends the proof. \square

Observe from the proof that we split the contour integral on \mathcal{E}_ρ into two parts on \mathcal{C}_ρ , which actually allows us to take the advantage of cancelation of $\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}$. Indeed, the bound (2.22) is tight, as we will see shortly that this argument can recover the best estimate for the Chebyshev case (see [31, Theorem 3.8] and (1.1)), and improve the bounds in [38] (see (1.2)).

2.3. Main results. For clarity of exposition, we first present the result on the general Jacobi polynomial expansions, followed by the refined results on the Chebyshev-type expansions ($\alpha = k - 1/2, \beta = l - 1/2$ with $k, l \in \mathbb{N} := \{0, 1, 2, \dots\}$), and Legendre-type expansions ($\alpha = k, \beta = l$ with $k, l \in \mathbb{N}$).

2.3.1. General Jacobi expansions ($\alpha, \beta > -1$).

Theorem 2.1. *For any $u \in \mathcal{A}_\rho$ (with $\rho > 1$), $\alpha, \beta > -1$ and $n \geq 0$, we have*

$$|\hat{u}_n^{\alpha,\beta}| \leq \frac{M}{\rho^n} \left[|\sigma_{n,0}^{\alpha,\beta}| + \frac{|\sigma_{n,1}^{\alpha,\beta}|}{\rho} + \frac{2}{\rho(\rho-1)} \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}} \right],\tag{2.25}$$

where

$$\sigma_{n,0}^{\alpha,\beta} = \frac{\sqrt{\pi}}{2} \frac{(2n+1)! \Gamma(n+\alpha+\beta+1)}{\Gamma(n+3/2) \Gamma(2n+\alpha+\beta+1)}, \quad \sigma_{n,1}^{\alpha,\beta} = \frac{(\beta-\alpha)(2n+2)}{2n+\alpha+\beta+2} \sigma_{n,0}^{\alpha,\beta},\tag{2.26}$$

and $\gamma_n^{\alpha,\beta}$ is defined in (2.8).

In particular, if $\alpha = \beta$, we have

$$|\hat{u}_n^{\alpha,\alpha}| \leq \frac{M}{\rho^n} \left[|\sigma_{n,0}^{\alpha,\alpha}| + \frac{2}{\rho^2-1} \sqrt{\frac{\gamma_0^{\alpha,\alpha}}{\gamma_n^{\alpha,\alpha}}} \right].\tag{2.27}$$

Proof. By (2.22),

$$|\hat{u}_n^{\alpha,\beta}| \leq \frac{M}{\rho^n} |\sigma_{n,0}^{\alpha,\beta}| + \frac{M}{\rho^{n+1}} |\sigma_{n,1}^{\alpha,\beta}| + \frac{M}{\rho^{n+2}} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \frac{1}{\rho^j}. \quad (2.28)$$

The factors $\sigma_{n,0}^{\alpha,\beta}$ and $\sigma_{n,1}^{\alpha,\beta}$ in (2.26) are computed from (2.21) directly, so it suffices to estimate the infinite sum in (2.28). Recall the identity (cf. [29]):

$$U_k(x) - U_{k-2}(x) = 2T_k(x), \quad k \geq 2. \quad (2.29)$$

Then we infer from (2.16) that

$$\begin{aligned} \sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta} &= \frac{1}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 (U_{n+j+2}(x) - U_{n+j}(x)) J_n^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) dx \\ &= \frac{2}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 T_{n+j+2}(x) J_n^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) dx, \quad n, j \geq 0. \end{aligned} \quad (2.30)$$

Thus, using the Cauchy-Schwartz inequality, the orthogonality (2.7), and the fact $|T_k(x)| \leq 1$, leads to

$$|\sigma_{n,j+2}^{\alpha,\beta} - \sigma_{n,j}^{\alpha,\beta}| \leq \frac{2}{\sqrt{\gamma_n^{\alpha,\beta}}} \left(\int_{-1}^1 T_{n+j+2}^2(x) \omega^{\alpha,\beta}(x) dx \right)^{1/2} \leq 2 \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}}. \quad (2.31)$$

Therefore, the bound (2.25) follows from $\sum_{j=0}^{\infty} \rho^{-j} = 1/(1 - \rho^{-1})$, as $\rho > 1$.

For $\alpha = \beta$, since $|\sigma_{n,2l+1}^{\alpha,\alpha}| = 0$, for all $l \geq 0$ (cf. Corollary 2.1 (i)), we have

$$\sum_{j=0}^{\infty} |\sigma_{n,j+2}^{\alpha,\alpha} - \sigma_{n,j}^{\alpha,\alpha}| \frac{1}{\rho^j} = \sum_{l=0}^{\infty} |\sigma_{n,2l+2}^{\alpha,\alpha} - \sigma_{n,2l}^{\alpha,\alpha}| \frac{1}{\rho^{2l}} \leq 2 \sqrt{\frac{\gamma_0^{\alpha,\alpha}}{\gamma_n^{\alpha,\alpha}}} \frac{1}{1 - \rho^{-2}}.$$

This yields the refined bound in (2.27). \square

Remark 2.2. Using Lemma 2.1, we can characterize the explicit dependence of the upper bounds in (2.25) and (2.27) on n, α, β . Indeed, for $\alpha, \beta > -1, n \geq 1$ and $n + \alpha + \beta > 0$,

$$\begin{aligned} \sigma_{n,0}^{\alpha,\beta} &\stackrel{(2.26)}{=} \frac{\sqrt{\pi}}{2} \frac{\Gamma(n + \alpha + \beta + 1)}{\Gamma(n + 3/2)} \frac{(2n + 1)!}{\Gamma(2n + \alpha + \beta + 1)} \\ &\stackrel{(2.10)}{\leq} \frac{\sqrt{\pi}}{2} (\Upsilon_n^{\alpha+\beta+1, 3/2} n^{\alpha+\beta+1-3/2}) (\Upsilon_{2n}^{2, \alpha+\beta+1} (2n)^{2-(\alpha+\beta+1)}) \\ &= \frac{\sqrt{\pi n}}{2^{\alpha+\beta}} \Upsilon_n^{\alpha+\beta+1, 3/2} \Upsilon_{2n}^{2, \alpha+\beta+1} \stackrel{(2.13)}{=} \frac{\sqrt{\pi n}}{2^{\alpha+\beta}} (1 + O(n^{-1})), \end{aligned} \quad (2.32)$$

which implies

$$\begin{aligned} |\sigma_{n,1}^{\alpha,\beta}| &\stackrel{(2.26)}{=} \frac{|\alpha - \beta|(2n + 2)}{2n + \alpha + \beta + 2} \sigma_{n,0}^{\alpha,\beta} \leq \frac{|\alpha - \beta|(2n + 2)}{2n + \alpha + \beta + 2} \frac{\sqrt{\pi n}}{2^{\alpha+\beta}} \Upsilon_n^{\alpha+\beta+1, 3/2} \Upsilon_{2n}^{2, \alpha+\beta+1} \\ &= |\alpha - \beta| \frac{\sqrt{\pi n}}{2^{\alpha+\beta}} (1 + O(n^{-1})). \end{aligned} \quad (2.33)$$

Similarly, one verifies

$$\begin{aligned}
\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}} &\stackrel{(2.8)}{=} (2n + \alpha + \beta + 1) \frac{\gamma_0^{\alpha,\beta}}{2^{\alpha+\beta+1}} \frac{n! \Gamma(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)} \\
&\leq (2n + \alpha + \beta + 1) \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \Upsilon_n^{1,\alpha+1} \Upsilon_n^{\alpha+\beta+1,\beta+1} \\
&\stackrel{(2.13)}{=} \frac{2\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} n(1 + O(n^{-1})).
\end{aligned} \tag{2.34}$$

Consequently, we infer from the estimate (2.25) that for fixed $\alpha, \beta > -1$ and $n \gg 1$,

$$|\hat{u}_n^{\alpha,\beta}| \leq C_n M \left(\frac{\sqrt{\pi}}{2^{\alpha+\beta}} \left(1 + \frac{|\alpha - \beta|}{\rho} \right) + \sqrt{\frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}} \frac{2\sqrt{2}}{\rho(\rho - 1)} \right) \frac{\sqrt{n}}{\rho^n}, \tag{2.35}$$

and likewise, we find from (2.27) that

$$|\hat{u}_n^{\alpha,\alpha}| \leq C_n M \left(\frac{\sqrt{\pi}}{2^{2\alpha}} + \frac{\Gamma(\alpha + 1)}{\sqrt{\Gamma(2\alpha + 2)}} \frac{2\sqrt{2}}{\rho^2 - 1} \right) \frac{\sqrt{n}}{\rho^n}, \tag{2.36}$$

where $C_n = 1 + O(n^{-1})$. \square

Remark 2.3. It is worthwhile to show that the bound obtained in this way is tighter than (1.2) obtained in [38]. Indeed, it follows from (A.5), (A.6b) and (2.7) that for $n \geq 1$ and $j = 0, 1$,

$$\begin{aligned}
\sigma_{n,j}^{\alpha,\beta} &= \frac{1}{\gamma_n^{\alpha,\beta}} \frac{1}{n + j + 1} \int_{-1}^1 T'_{n+j+1}(x) J_n^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) dx \\
&= \frac{2}{\gamma_n^{\alpha,\beta}} \sum_{\substack{k=0 \\ k+n+j+1 \text{ odd}}}^{n+j} \frac{1}{c_k} \int_{-1}^1 T_k(x) J_n^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) dx \\
&= \frac{2}{\gamma_n^{\alpha,\beta}} \int_{-1}^1 T_{n+j}(x) J_n^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) dx,
\end{aligned}$$

where $c_0 = 2$ and $c_k = 1$ for $k \geq 1$. Following (2.30)-(2.31), we have

$$|\sigma_{n,j}^{\alpha,\beta}| \leq 2 \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}}, \quad n \geq 1, \quad j = 0, 1.$$

Finally, a straightforward calculation leads to

$$\frac{M}{\rho^n} \left[|\sigma_{n,0}^{\alpha,\beta}| + \frac{|\sigma_{n,1}^{\alpha,\beta}|}{\rho} + \frac{2}{\rho(\rho - 1)} \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}} \right] \leq \frac{2M}{\rho^{n-1}(\rho - 1)} \sqrt{\frac{\gamma_0^{\alpha,\beta}}{\gamma_n^{\alpha,\beta}}}. \tag{2.37}$$

Moreover, we claim from (2.27) that the strict inequality holds, when $\alpha = \beta > -1$. One may refer to Section 4 for numerical evidences. \square

2.3.2. Chebyshev-type expansions ($\alpha = k - 1/2, \beta = l - 1/2$ with $k, l \in \mathbb{N}$).

In view of (2.19), it follows from (2.23) that the Chebyshev coefficient takes the simplest form:

$$\hat{u}_n^{-1/2,-1/2} = \frac{\sigma_{n,0}^{-1/2,-1/2}}{2\pi i} \oint_{\mathcal{C}_\rho} \frac{u(z)}{w^{n+1}} dw. \tag{2.38}$$

Thus, using (2.19) and (2.22) leads to

$$|\hat{u}_n^{-1/2, -1/2}| \leq \frac{2\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+1/2)} \frac{M}{\rho^n}. \quad (2.39)$$

This leads to the estimate for the expansion coefficients, denoted by $\{\hat{u}_n^C\}$ as before, in terms of $\{T_n(x)\}$:

$$|\hat{u}_n^C| \leq \frac{2M}{\rho^n}, \quad n \geq 0, \quad (2.40)$$

as documented in e.g., [31].

For the second-kind Chebyshev case, we find from (2.15) the closed-form formula like (2.38):

$$\hat{u}_n^{1/2, 1/2} = \frac{\sigma_{n,0}^{1/2, 1/2}}{\pi i} \oint_{\mathcal{E}_\rho} \frac{u(z)}{w^{n+1}} dz, \quad (2.41)$$

but the contour integration is on \mathcal{E}_ρ . It follows from (2.18) and (2.23) that

$$|\hat{u}_n^{1/2, 1/2}| \leq \frac{1}{2\sqrt{\pi}} \frac{(n+1)!}{\Gamma(n+3/2)} \left| \oint_{\mathcal{E}_\rho} \frac{u(z)}{w^{n+1}} dz \right| \leq \frac{\sqrt{\pi}}{2} \frac{(n+1)!}{\Gamma(n+3/2)} \frac{M}{\rho^n} \left(1 + \frac{1}{\rho^2}\right). \quad (2.42)$$

Like (2.40), if we re-scale the expansion in terms of $\{U_n\}$, i.e.,

$$\hat{u}_n^U = \frac{2}{\pi} \int_{-1}^1 u(x) U_n(x) \sqrt{1-x^2} dx,$$

then we find from (A.5) and (2.42) that

$$|\hat{u}_n^U| = \frac{2}{\sqrt{\pi}} \frac{\Gamma(n+3/2)}{\Gamma(n+2)} |\hat{u}_n^{1/2, 1/2}| \leq \frac{M}{\rho^n} \left(1 + \frac{1}{\rho^2}\right). \quad (2.43)$$

Remark 2.4. It is seen from (2.41) that the second-kind Chebyshev coefficient takes the simplest form on the contour \mathcal{E}_ρ . This motivates us to estimate the contour integral directly by

$$\left| \oint_{\mathcal{E}_\rho} \frac{u(z)}{w^{n+1}} dz \right| \leq \frac{M}{\rho^{n+1}} \oint_{\mathcal{E}_\rho} |dz| = \frac{M}{\rho^{n+1}} L(\mathcal{E}_\rho),$$

which implies

$$|\hat{u}_n^U| \leq \frac{M}{\rho^{n+1}} \frac{L(\mathcal{E}_\rho)}{\pi}. \quad (2.44)$$

By (2.4),

$$\frac{L(\mathcal{E}_\rho)}{\pi \rho} \leq \sqrt{1 + \frac{1}{\rho^4}} < 1 + \frac{1}{\rho^2}.$$

Therefore, the estimate (2.44) is slightly sharper than (2.43). \square

Some refined results can also be derived for $\alpha = k+1/2, \beta = l+1/2$ with $k, l \in \mathbb{N}$. Indeed, we find that $\{\sigma_{n,j}^{k+1/2, l+1/2}\}$ can be computed explicitly by the following formula.

Proposition 2.1. *For any $k, l, n, j \in \mathbb{N}$,*

$$\sigma_{n,j}^{k+1/2, l+1/2} = \sqrt{\frac{\pi}{2}} \frac{1}{\gamma_n^{k+1/2, l+1/2}} \sum_{m=n}^{n+k+l} d_m^{k+1/2, l+1/2} \sqrt{\gamma_m^{1/2, 1/2}} \delta_{m, n+j}, \quad (2.45)$$

where $\{d_m^{k+1/2, l+1/2}\}_{m=n}^{n+k+l}$ are given in (A.3), and $\delta_{m, n+j}$ is the Kronecker delta.

Proof. Using (A.3) (with $\alpha = \beta = 1/2$), (2.16) and the properties of Jacobi polynomials (cf. (2.7) and (A.5)), leads to

$$\begin{aligned}\sigma_{n,j}^{k+1/2,l+1/2} &= \frac{1}{\gamma_n^{k+1/2,l+1/2}} \sum_{m=n}^{n+k+l} d_m^{k+1/2,l+1/2} \int_{-1}^1 U_{n+j}(x) J_m^{1/2,1/2}(x) (1-x^2)^{1/2} dx \\ &= \frac{1}{\gamma_n^{k+1/2,l+1/2}} \sqrt{\frac{\pi}{2}} \sum_{m=n}^{n+k+l} d_m^{k+1/2,l+1/2} \sqrt{\gamma_m^{1/2,1/2}} \delta_{m,n+j},\end{aligned}\quad (2.46)$$

This completes the proof. \square

Equipped with (2.45), we can obtain the bound for Chebyshev-type expansion coefficients by computing $\{d_m^{k+1/2,l+1/2}\}$ explicitly. To fix the idea, we just consider the case: $k = 1$ and $l = 0$. One finds

$$d_n^{3/2,1/2} = 1, \quad d_{n+1}^{3/2,1/2} = -\frac{2n+2}{2n+3},$$

and

$$\sigma_{n,0}^{3/2,1/2} = \frac{\sqrt{\pi}}{4} \frac{n!(n+2)}{\Gamma(n+3/2)}, \quad \sigma_{n,1}^{3/2,1/2} = -\frac{\sqrt{\pi}}{4} \frac{(n+1)!}{\Gamma(n+3/2)}, \quad \sigma_{n,j}^{3/2,1/2} = 0, \quad j \geq 2.$$

The estimate (2.14) reduces to

$$\begin{aligned}\hat{u}_n^{3/2,1/2} &\leq \frac{M}{\rho^n} \left[\sigma_{n,0}^{3/2,1/2} + \frac{\sigma_{n,1}^{3/2,1/2}}{\rho} + \frac{\sigma_{n,0}^{3/2,1/2}}{\rho^2} + \frac{\sigma_{n,1}^{3/2,1/2}}{\rho^3} \right] \\ &= \frac{M}{\rho^n} \left(1 + \frac{1}{\rho^2} \right) \left[\sigma_{n,0}^{3/2,1/2} + \frac{\sigma_{n,1}^{3/2,1/2}}{\rho} \right].\end{aligned}$$

Thus, we have

$$|\hat{u}_n^{3/2,1/2}| \leq \frac{\sqrt{\pi}}{4} \frac{(n+1)!}{\Gamma(n+3/2)} \frac{M}{\rho^n} \left(1 + \frac{1}{\rho^2} \right) \left(\frac{n+2}{n+1} + \frac{1}{\rho} \right), \quad (2.47)$$

and by (2.10), we have for $n \geq 0$,

$$\frac{(n+1)!}{\Gamma(n+3/2)} \leq \sqrt{n} \exp \left(\frac{8n+7}{12(2n+1)(n+1)} + \frac{1}{4n} \right). \quad (2.48)$$

Actually, the infinite sum in (2.22) does not appear for the Chebyshev-type expansions, which allows us to derive very tight bounds. However, for the Legendre-type expansions, some care has to be taken to handle this sum.

2.3.3. Legendre-type expansions ($\alpha = k, \beta = l$ with $k, l \in \mathbb{N}$).

We first consider the Legendre case. By (2.8) and (2.26),

$$\gamma_n^{0,0} = \frac{2}{2n+1}, \quad \frac{\gamma_0^{0,0}}{\gamma_n^{0,0}} = 2n+1, \quad \sigma_{n,0}^{0,0} = \frac{\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+1/2)},$$

so the estimate (2.27) reduces to

$$|\hat{u}_n^{0,0}| \leq \frac{M}{\rho^n} \left[\frac{\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+1/2)} + \frac{2\sqrt{2n+1}}{\rho^2-1} \right]. \quad (2.49)$$

In fact, we can improve this estimate, as highlighted in the following theorem, by using the explicit information of $\sigma_{n,2l}^{0,0}$.

Theorem 2.2. *Let $\{\hat{u}_n^{0,0}\}$ be the Legendre expansion coefficients of any $u \in \mathcal{A}_\rho$ with $\rho > 1$. Then for any $n \geq 1$,*

$$|\hat{u}_n^{0,0}| \leq \frac{M\sqrt{\pi n}}{\rho^n} \left(1 + \frac{n+2}{2n+3} \frac{1}{\rho^2-1}\right) \exp\left(\frac{8n-1}{12n(2n-1)}\right). \quad (2.50)$$

Proof. A straightforward calculation from (2.20) yields

$$\sigma_{n,2l+2}^{0,0} - \sigma_{n,2l}^{0,0} = -\frac{n+2l+2}{2(l+1)(n+l+3/2)} \sigma_{n,2l}^{0,0}, \quad l \geq 0, \quad (2.51)$$

which implies $\{\sigma_{n,2l}^{0,0}\}$ is strictly descending with respect to l . Hence, we have

$$|\sigma_{n,2l+2}^{0,0} - \sigma_{n,2l}^{0,0}| = \frac{n+2l+2}{2(l+1)(n+l+3/2)} \sigma_{n,2l}^{0,0} \leq \frac{n+2}{2n+3} \sigma_{n,0}^{0,0}, \quad (2.52)$$

where we used the fact that $n+2l+2/((l+1)(n+l+3/2))$ is strictly descending with respect to l . Then, we obtain the improved bound from (2.22):

$$|\hat{u}_n^{0,0}| \leq \frac{M}{\rho^n} \sigma_{n,0}^{0,0} \left(1 + \frac{n+2}{2n+3} \sum_{l=0}^{\infty} \frac{1}{\rho^{2l+2}}\right) = \frac{\sqrt{\pi}M}{\rho^n} \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \left(1 + \frac{n+2}{2n+3} \frac{1}{\rho^2-1}\right), \quad (2.53)$$

and by (2.10),

$$\frac{\Gamma(n+1)}{\Gamma(n+1/2)} \leq \sqrt{n} \exp\left(\frac{8n-1}{12n(2n-1)}\right), \quad n \geq 1. \quad (2.54)$$

This completes the proof. \square

Remark 2.5. We compare the bound in (2.50) with the existing ones. Davis [12, Page 313] stated the bound

$$|\hat{u}_n^{0,0}| \leq \frac{2n+1}{2} \frac{ML(\mathcal{E}_\rho)}{\rho^n(\rho-1)} \stackrel{(2.4)}{\leq} \frac{2n+1}{2} \frac{\pi\sqrt{\rho^2+\rho^{-2}}M}{\rho^n(\rho-1)},$$

where clearly the algebraic order of n in the numerator is not optimal. The following asymptotic bound can be obtained from [27, Eq. (32) and Eq. (38)] and [12, Eq. (12.4.25)]:

$$|\hat{u}_n^{0,0}| \leq \frac{M\sqrt{\pi n}}{\rho^n} \frac{\sqrt{\rho^4+1}}{\rho^2-1}, \quad n \gg 1,$$

while the asymptotic estimate derived from (2.50) is

$$|\hat{u}_n^{0,0}| \leq \frac{M\sqrt{\pi n}}{\rho^n} \frac{\rho^2-1/2}{\rho^2-1}, \quad n \gg 1, \quad (2.55)$$

which is sharper. Another bound for comparison is obtained in the recent paper [38]:

$$|\hat{u}_n^{0,0}| \leq \frac{2\sqrt{n}M}{\rho^n} \left(1 + \frac{1}{\rho^2-1}\right), \quad n \geq 1, \quad (2.56)$$

which is also inferior to our estimate (2.50). Some comparisons in numerical perspective are given in Section 4. \square

Like the Chebyshev case, we can derive similar refined estimates for Legendre-type expansions with $\alpha = k, \beta = l$ and $k, l \in \mathbb{N}$. The counterpart of Proposition 2.1 is stated as follows, which can be obtained by using (A.3) (with $\alpha = \beta = 0$), (2.16) and the properties of Jacobi polynomials (e.g., (2.7)) as before.

Proposition 2.2. *For any $k, l, n, j \in \mathbb{N}$,*

$$\sigma_{n,j}^{k,l} = \frac{1}{\gamma_n^{k,l}} \sum_{m=n}^{n+k+l} d_m^{k,l} \gamma_m^{0,0} \sigma_{m,n+j-m}^{0,0}, \quad (2.57)$$

where $\{d_m^{k,l}\}_{m=n}^{n+k+l}$ are the same as in (A.3), and $\{\sigma_{m,n+j-m}^{0,0}\}$ are computed by (2.20).

Once again, to fix the idea, we just consider the case: $k = 1$ and $l = 0$. One finds $d_n^{1,0} = 1, d_{n+1}^{1,0} = -1$, and

$$\sigma_{n,j}^{1,0} = \frac{1}{\gamma_n^{1,0}} (\gamma_n^{0,0} \sigma_{n,j}^{0,0} - \gamma_{n+1}^{0,0} \sigma_{n+1,j-1}^{0,0}) = \frac{n+1}{2n+1} \sigma_{n,j}^{0,0} - \frac{n+1}{2n+3} \sigma_{n+1,j-1}^{0,0}.$$

By (2.20),

$$\sigma_{n,2l}^{1,0} = \frac{n+1}{2n+1} \sigma_{n,2l}^{0,0}, \quad \sigma_{n,2l+1}^{1,0} = -\frac{n+1}{2n+3} \sigma_{n+1,2l}^{0,0}, \quad l \geq 0.$$

Therefore, with (2.51) and (2.52), the estimate (2.22) reduces to

$$\begin{aligned} |\hat{u}_n^{1,0}| &\leq \frac{M}{\rho^n} |\sigma_{n,0}^{1,0}| + \frac{M}{\rho^{n+1}} |\sigma_{n,1}^{1,0}| + \frac{M}{\rho^{n+2}} \sum_{j=0}^{\infty} |\sigma_{n,j+2}^{1,0} - \sigma_{n,j}^{1,0}| \frac{1}{\rho^j} \\ &= \frac{M}{\rho^n} \frac{n+1}{2n+1} \left(\sigma_{n,0}^{0,0} + \frac{1}{\rho^2} \sum_{l=0}^{\infty} |\sigma_{n,2l+2}^{0,0} - \sigma_{n,2l}^{0,0}| \frac{1}{\rho^{2l}} \right) \\ &\quad + \frac{M}{\rho^{n+1}} \frac{n+1}{2n+3} \left(\sigma_{n+1,0}^{0,0} + \frac{1}{\rho^2} \sum_{l=0}^{\infty} |\sigma_{n+1,2l+2}^{0,0} - \sigma_{n+1,2l}^{0,0}| \frac{1}{\rho^{2l}} \right) \\ &\leq \sigma_{n,0}^{0,0} \frac{n+1}{2n+1} \frac{M}{\rho^n} \left(1 + \frac{n+2}{2n+3} \frac{1}{\rho^2 - 1} \right) + \sigma_{n+1,0}^{0,0} \frac{n+1}{2n+3} \frac{M}{\rho^{n+1}} \left(1 + \frac{n+3}{2n+5} \frac{1}{\rho^2 - 1} \right). \end{aligned}$$

Working out the expressions of $\sigma_{n,0}^{0,0}$ and $\sigma_{n+1,0}^{0,0}$ by (2.26), we have

$$|\hat{u}_n^{1,0}| \leq \frac{M}{\rho^n} \frac{\sqrt{\pi} \Gamma(n+2)}{\Gamma(n+3/2)} \left\{ \frac{1}{2} + \frac{n+2}{2(2n+3)} \frac{1}{\rho^2 - 1} + \frac{1}{\rho} \frac{n+1}{2n+3} \left(1 + \frac{n+3}{2n+5} \frac{1}{\rho^2 - 1} \right) \right\}. \quad (2.58)$$

Note that the ratio of the Gamma functions can be bounded as in (2.48).

The same process applies to other $k, l \in \mathbb{N}$, but the derivation seems tedious.

2.4. Estimates for truncated Jacobi expansions. Given a cut-off number $N \geq 1$ and $N \in \mathbb{N}$, we define the partial sum

$$(\pi_N^{\alpha,\beta} u)(x) = \sum_{n=0}^{N-1} \hat{u}_n^{\alpha,\beta} J_n^{\alpha,\beta}(x), \quad (2.59)$$

where $\{\hat{u}_n^{\alpha,\beta}\}$ are the Jacobi expansion coefficients defined in (2.14). To this end, let $L_{\omega^{\alpha,\beta}}^2(I)$ be the weighted L^2 -space on $I = (-1, 1)$, and its norm is denoted by $\|\cdot\|_{\omega^{\alpha,\beta}}$, where we drop the weight function, if $\alpha = \beta = 0$.

Notice that $\pi_N^{\alpha,\beta} u$ is the $L_{\omega^{\alpha,\beta}}^2$ -projection of u upon P_{N-1} (denoting the set of all algebraic polynomials of degree at most $N-1$), that is, $\pi_N^{\alpha,\beta} u$ is the best approximation to u in the norm $\|\cdot\|_{\omega^{\alpha,\beta}}$. With the previous bounds for the expansion coefficients, we can estimate the truncation error straightforwardly.

Theorem 2.3. For any $u \in \mathcal{A}_\rho$ with $\rho > 1$, and $\alpha, \beta > -1$, we have

$$\|\pi_N^{\alpha,\beta} u - u\|_{\omega^{\alpha,\beta}} \leq \left[\sqrt{\frac{\pi}{2^{\alpha+\beta}}} \left(1 + \frac{|\alpha - \beta|}{\rho} \right) + \frac{2\sqrt{\gamma_0^{\alpha,\beta}}}{\rho(\rho-1)} \right] \frac{C_N M}{\rho^{N-1} \sqrt{\rho^2 - 1}}, \quad (2.60)$$

where $\gamma_0^{\alpha,\beta}$ is given in (2.8) and $C_N \approx 1$.

Proof. By the orthogonality (cf (2.7)-(2.8)) of Jacobi polynomials, we have

$$\|\pi_N^{\alpha,\beta} u - u\|_{\omega^{\alpha,\beta}}^2 = \sum_{n=N}^{\infty} |\hat{u}_n^{\alpha,\beta}|^2 \gamma_n^{\alpha,\beta}.$$

It follows from the estimate of $|\hat{u}_n^{\alpha,\beta}|$ in Theorem 2.1, and a combination of (2.12)-(2.13) and (2.32)-(2.33) that for $n \geq N \gg 1$,

$$\begin{aligned} |\hat{u}_n^{\alpha,\beta}| \sqrt{\gamma_n^{\alpha,\beta}} &\leq \frac{M}{\rho^n} \left[|\sigma_{n,0}^{\alpha,\beta}| \sqrt{\gamma_n^{\alpha,\beta}} + \frac{1}{\rho} |\sigma_{n,1}^{\alpha,\beta}| \sqrt{\gamma_n^{\alpha,\beta}} + \frac{2}{\rho(\rho-1)} \sqrt{\gamma_0^{\alpha,\beta}} \right] \\ &\leq \frac{C_n M}{\rho^n} \left[\sqrt{\frac{\pi}{2^{\alpha+\beta}}} \left(1 + \frac{|\alpha - \beta|}{\rho} \right) + \frac{2}{\rho(\rho-1)} \sqrt{\gamma_0^{\alpha,\beta}} \right], \end{aligned}$$

where $C_n = 1 + O(n^{-1})$. Therefore, we have

$$\begin{aligned} \|\pi_N^{\alpha,\beta} u - u\|_{\omega^{\alpha,\beta}} &\leq C_N M \left[\sqrt{\frac{\pi}{2^{\alpha+\beta}}} \left(1 + \frac{|\alpha - \beta|}{\rho} \right) + \frac{2\sqrt{\gamma_0^{\alpha,\beta}}}{\rho(\rho-1)} \right] \left(\sum_{n=N}^{\infty} \frac{1}{\rho^{2n}} \right)^{1/2} \\ &\leq \left[\sqrt{\frac{\pi}{2^{\alpha+\beta}}} \left(1 + \frac{|\alpha - \beta|}{\rho} \right) + \frac{2\sqrt{\gamma_0^{\alpha,\beta}}}{\rho(\rho-1)} \right] \frac{C_N M}{\rho^{N-1} \sqrt{\rho^2 - 1}}. \end{aligned}$$

This ends the proof. \square

Remark 2.6. Note that $\{\frac{d^l}{dx^l} J_n^{\alpha,\beta}\}_{n \geq l}$ are mutually orthogonal with respect to $\omega^{\alpha+l, \beta+l}$, so we can estimate $\|(\pi_N^{\alpha,\beta} u - u)^{(l)}\|_{\omega^{\alpha+l, \beta+l}}$ in a similar fashion. \square

Remark 2.7. Some refined estimates can be obtained from the refined bounds for special cases, e.g., $\alpha = \beta$ or $\alpha = \beta = 0, -1/2$. Here, we just state the result for the Legendre case:

$$\|\pi_N^{0,0} u - u\| \leq \left(1 + \frac{1}{2(\rho^2 - 1)} \right) \frac{C_N \sqrt{\pi} M}{\rho^{N-1} \sqrt{\rho^2 - 1}}, \quad (2.61)$$

where $C_N \approx 1$ as before. It follows from Theorem 2.2 and the above process. Note that Xiang [38] derived the following estimate for the Legendre expansion:

$$\|\pi_N^{0,0} u - u\| \leq \frac{2\sqrt{2}M}{\rho^{N-2}(\rho-1)^2}. \quad (2.62)$$

The estimate (2.61) seems tighter than this one. \square

3. ERROR ESTIMATES FOR GEGENBAUER-GAUSS QUADRATURE

3.1. Preliminaries. The Gegenbauer-Gauss quadrature remainder (1.3)-(1.4) with the nodes being zeros of the Gegenbauer polynomial $J_n^{\alpha,\alpha}(x)$, takes the form

$$E_n^{GG}[u] = \frac{\gamma_n^{\alpha,\alpha}}{\pi i} \oint_{\mathcal{E}_\rho} \frac{Q_n^{\alpha,\alpha}(z)}{J_n^{\alpha,\alpha}(z)} u(z) dz, \quad \forall u \in \mathcal{A}_\rho, \quad (3.1)$$

where $Q_n^{\alpha,\alpha}(z)$ is defined as in (B.2), namely,

$$Q_n^{\alpha,\alpha}(z) = \frac{1}{2\gamma_n^{\alpha,\alpha}} \int_{-1}^1 \frac{J_n^{\alpha,\alpha}(x)\omega^{\alpha,\alpha}(x)}{z-x} dx \stackrel{(B.7)}{=} \sum_{j=0}^{\infty} \frac{\sigma_{n,j}^{\alpha,\alpha}}{w^{n+j+1}} \stackrel{(2.17)}{=} \sum_{l=0}^{\infty} \frac{\sigma_{n,2l}^{\alpha,\alpha}}{w^{n+2l+1}}. \quad (3.2)$$

As already mentioned, the analysis of quadrature errors (even for the Chebyshev case) has attracted much attention (see e.g., [11, 10, 3, 17, 13, 16, 25, 26]). Just to mention that Chawla and Jain [11, Theorem 5] obtained the estimate:

$$|E_n^{CG}[u]| \leq \frac{2\pi M}{\rho^{2n}-1}, \quad \forall u \in \mathcal{A}_\rho, \quad \forall n \geq 1, \quad (3.3)$$

Hunter [25] derived the general bound

$$|E_n^{GG}[u]| \leq \frac{4 \int_{-1}^1 (1-x^2)^\alpha dx}{\rho^{2n-2}(\rho^2-1)}, \quad n \geq 1, \quad (3.4)$$

and some refined results for $\alpha = \pm 1/2$ and $\beta = \pm 1/2$ by expanding $Q_n^{\alpha,\alpha}/J_n^{\alpha,\alpha}$ into the Laurent series of w in the disk enclosed by \mathcal{C}_ρ , and manipulating the series. It is worthwhile to note that Gautschi and Varga [17] estimated the Jacobi-Gauss quadrature (with $J_n^{\alpha,\beta}$ and $Q_n^{\alpha,\beta}$ in place of $J_n^{\alpha,\alpha}$ and $Q_n^{\alpha,\alpha}$ in (3.1), respectively) by

$$|E_n^{JG}[u]| \leq \pi^{-1} \gamma_n^{\alpha,\beta} ML(\mathcal{E}_\rho) \max_{z \in \mathcal{E}_\rho} |Q_n^{\alpha,\beta}(z)/J_n^{\alpha,\beta}(z)|, \quad (3.5)$$

and attempted to find the exact maximum value on the Bernstein ellipse, which was feasible for $\alpha = \pm 1/2$ and $\beta = \pm 1/2$ again. Some conjectures and empirical results were explored in [17] for the general Jacobi case.

Using the explicit expression of Legendre polynomials on the Bernstein ellipse (see e.g., [12, Lemma 12.4.1]), Kambo [27] obtained the bound for the Legendre-Gauss quadrature:

$$|E_n^{LG}[u]| \leq \pi^{-1} \gamma_n^{0,0} ML(\mathcal{E}_\rho) \frac{\max_{z \in \mathcal{E}_\rho} |Q_n^{0,0}(z)|}{\min_{z \in \mathcal{E}_\rho} |J_n^{0,0}(z)|} \leq \frac{d_n M}{\rho^{2n}} \frac{\rho^2 + 1}{\rho^2 - 2}, \quad \rho > \sqrt{2}, \quad (3.6)$$

where $0 < d_n \leq \pi$. While this bound is only valid for $\rho > \sqrt{2}$, it holds for all n , when compared with the asymptotic estimate (with $n \gg 1$) for the Legendre-Gauss quadrature in [9].

In what follows, we aim to extend our analysis to estimate $E_n^{GG}[u]$ in (3.1). The essential tools include the explicit formula for the Gegenbauer polynomial $J_n^{\alpha,\alpha}(z)$ on \mathcal{E}_ρ derived in our recent paper [39], and the previous argument for estimating $Q_n^{\alpha,\alpha}(z)$. Let us recall the important formula stated in [39, Lemma 3.1].

Lemma 3.1. *Let $z = \frac{1}{2}(w + w^{-1})$. Then we have*

$$J_n^{\alpha,\alpha}(z) = A_n^\alpha \sum_{k=0}^n g_k^\alpha g_{n-k}^\alpha w^{n-2k}, \quad n \geq 0, \quad \alpha > -1, \quad \alpha \neq -1/2, \quad (3.7)$$

where

$$g_0^\alpha = 1, \quad g_k^\alpha = \frac{\Gamma(k + \alpha + 1/2)}{k! \Gamma(\alpha + 1/2)}, \quad 1 \leq k \leq n, \quad \text{and} \quad A_n^\alpha = \frac{\Gamma(2\alpha + 1) \Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1) \Gamma(n + 2\alpha + 1)}. \quad (3.8)$$

Remark 3.1. This formula excludes the Chebyshev case. For $\alpha = -1/2$, we define

$$g_0^{-1/2} = g_n^{-1/2} = 1, \quad g_k^{-1/2} = 0, \quad 1 \leq k \leq n-1, \quad \text{and} \quad A_n^{-1/2} = \frac{\Gamma(n + 1/2)}{2\sqrt{\pi}n!}, \quad (3.9)$$

since (see e.g., [12])

$$T_n(z) = \frac{1}{2}(w^n + w^{-n}) = \frac{1}{2A_n^{-1/2}} J_n^{-1/2, -1/2}(z). \quad (3.10)$$

Hence, we understand that (3.7) holds for $\alpha = -1/2$ with the constants given by (3.9). \square

3.2. Main results. We adopt two approaches to estimate the quadrature remainder. The first one is to expand $Q_n^{\alpha, \alpha}/J_n^{\alpha, \alpha}$ in Laurent series of $w \in \mathcal{C}_\rho$, and then we use an argument as for Theorem 2.3 to obtain the tight error bound. However, this situation is reminiscent to that in Gautschi and Varga [17], that is, computable bounds can be derived for general α . We highlight that the computational part (see (3.11)) is independent of ρ and u .

The second approach is based on an important relation between the quadrature remainder and Gegenbauer expansion coefficient (see (3.22)).

The main estimate resulted from the first approach is stated as follows.

Theorem 3.1. *For any $u \in \mathcal{A}_\rho$ with $\rho > 1$, we have that for $\alpha > -1$ and $n \geq 1$,*

$$|E_n^{GG}[u]| \leq \gamma_n^{\alpha, \alpha} \left[|\mu_{n,0}^{\alpha, \alpha}| + \max_{l \geq 0} |\mu_{n,2l+2}^{\alpha, \alpha} - \mu_{n,2l}^{\alpha, \alpha}| \frac{1}{\rho^2 - 1} \right] \frac{M}{\rho^{2n}}, \quad (3.11)$$

where $\{\mu_{n,2l}^{\alpha, \alpha}\}_{l \geq 0}$ are computed by the recursive formula:

$$\mu_{n,2l}^{\alpha, \alpha} = \frac{1}{g_n^\alpha} \left(\frac{\sigma_{n,2l}^{\alpha, \alpha}}{A_n^\alpha} - \sum_{k=1}^{\min\{n,l\}} g_k^\alpha g_{n-k}^\alpha \mu_{n,2l-2k}^{\alpha, \alpha} \right), \quad l \geq 1, \quad \mu_{n,0}^{\alpha, \alpha} = \frac{\sigma_{n,0}^{\alpha, \alpha}}{A_n^\alpha g_n^\alpha}. \quad (3.12)$$

Proof. A straightforward calculation from (3.2) (note: $\sigma_{n,2l+1}^{\alpha, \alpha} = 0$ for all $l \geq 0$) and (3.7) leads to

$$\frac{Q_n^{\alpha, \alpha}(z)}{J_n^{\alpha, \alpha}(z)} = \sum_{l=0}^{\infty} \frac{\mu_{n,2l}^{\alpha, \alpha}}{w^{2n+2l+1}} \quad \text{with} \quad \sigma_{n,2l}^{\alpha, \alpha} = A_n^\alpha \sum_{k=0}^{\min\{n,l\}} g_k^\alpha g_{n-k}^\alpha \mu_{n,2l-2k}^{\alpha, \alpha}, \quad (3.13)$$

so solving out $\mu_{n,2l}^{\alpha, \alpha}$ yields (3.12).

Next, following the same lines as the derivation of (2.23), we infer from (3.1) and (3.13) that

$$\begin{aligned} |E_n^{GG}[u]| &\leq \gamma_n^{\alpha, \alpha} \frac{M}{2\pi} \left| \sum_{l=0}^{\infty} \mu_{n,2l}^{\alpha, \alpha} \oint_{\mathcal{C}_\rho} \frac{1}{w^{2n+2l+1}} \left(1 - \frac{1}{w^2}\right) dw \right| \\ &\leq \gamma_n^{\alpha, \alpha} \frac{M}{2\pi} \left[\frac{2\pi\rho}{\rho^{2n+1}} |\mu_{n,0}^{\alpha, \alpha}| + \frac{2\pi\rho}{\rho^{2n+3}} \sum_{l=0}^{\infty} |\mu_{n,2l+2}^{\alpha, \alpha} - \mu_{n,2l}^{\alpha, \alpha}| \frac{1}{\rho^{2l}} \right] \\ &= \gamma_n^{\alpha, \alpha} \frac{M}{\rho^{2n}} \left[|\mu_{n,0}^{\alpha, \alpha}| + \frac{1}{\rho^2} \sum_{l=0}^{\infty} |\mu_{n,2l+2}^{\alpha, \alpha} - \mu_{n,2l}^{\alpha, \alpha}| \frac{1}{\rho^{2l}} \right] \\ &\leq \gamma_n^{\alpha, \alpha} \frac{M}{\rho^{2n}} \left[|\mu_{n,0}^{\alpha, \alpha}| + \max_{l \geq 0} |\mu_{n,2l+2}^{\alpha, \alpha} - \mu_{n,2l}^{\alpha, \alpha}| \sum_{l=0}^{\infty} \frac{1}{\rho^{2l+2}} \right] \\ &= \gamma_n^{\alpha, \alpha} \frac{M}{\rho^{2n}} \left[|\mu_{n,0}^{\alpha, \alpha}| + \max_{l \geq 0} |\mu_{n,2l+2}^{\alpha, \alpha} - \mu_{n,2l}^{\alpha, \alpha}| \frac{1}{\rho^2 - 1} \right]. \end{aligned} \quad (3.14)$$

This completes the proof. \square

Remark 3.2. We find from (3.12) that for $\alpha = -1/2$,

$$\mu_{n,0}^{-1/2,-1/2} = \frac{2\pi}{\gamma_n^{-1/2,-1/2}}, \quad |\mu_{n,2l+2}^{-1/2,-1/2} - \mu_{n,2l}^{-1/2,-1/2}| = \frac{2\pi\delta_{\kappa,0}}{\gamma_n^{-1/2,-1/2}}, \quad \kappa := \text{mod}(l+1, n),$$

where $\delta_{\kappa,0}$ is the Kronecker delta. Hence, it follows from (3.14) that

$$|E_n^{CG}[u]| \leq \frac{2\pi M}{\rho^{2n}} \left[1 + \frac{1}{\rho^2} \sum_{j=1}^{\infty} \frac{1}{\rho^{2(jn-1)}} \right] = \frac{2\pi M}{\rho^{2n}-1}, \quad n \geq 1, \quad (3.15)$$

which is the same as (3.3) derived in [11]. \square

Remark 3.3. We find from (3.12) that for $\alpha = 1/2$,

$$\mu_{n,2l}^{1/2,1/2} = \begin{cases} (-1)^\kappa \frac{\pi}{2} \frac{1}{\gamma_n^{1/2,1/2}}, & \text{if } \kappa := \text{mod}(l, n+1) = 0, 1, \\ 0, & \text{otherwise,} \end{cases} \quad (3.16)$$

which implies

$$\begin{aligned} \sum_{l=0}^{\infty} |\mu_{n,2l+2}^{1/2,1/2} - \mu_{n,2l}^{1/2,1/2}| \frac{1}{\rho^{2l}} &= |\mu_{n,2}^{1/2,1/2} - \mu_{n,0}^{1/2,1/2}| + \sum_{j=1}^{\infty} |\mu_{n,2j(n+1)}^{1/2,1/2}| \frac{1}{\rho^{2j(n+1)-2}} \\ &+ \sum_{j=1}^{\infty} \left(|\mu_{n,2j(n+1)+2}^{1/2,1/2} - \mu_{n,2j(n+1)}^{1/2,1/2}| \frac{1}{\rho^{2j(n+1)}} + |\mu_{n,2j(n+1)+2}^{1/2,1/2}| \frac{1}{\rho^{2j(n+1)+2}} \right) \\ &= \frac{\pi}{2} \frac{1}{\gamma_n^{1/2,1/2}} \left(2 + (\rho + \rho^{-1})^2 \sum_{j=1}^{\infty} \frac{1}{\rho^{2j(n+1)}} \right) = \frac{\pi}{2} \frac{1}{\gamma_n^{1/2,1/2}} \left(2 + \frac{(\rho + \rho^{-1})^2}{\rho^{2n+2}-1} \right). \end{aligned}$$

Hence, it follows from (3.14) that for the Chebyshev-Gauss quadrature of the second kind,

$$|E_n^{GG}[u]| \leq \frac{\pi M}{2\rho^{2n}} \left(1 + \frac{1}{\rho^2} \left(2 + \frac{(\rho + \rho^{-1})^2}{\rho^{2n+2}-1} \right) \right) = \frac{\pi M(\rho^2 + 2 + \rho^{-2n-4})}{2(\rho^{2n+2}-1)}. \quad (3.17)$$

Note that Hunter [25, (4.8)] obtained the following estimate by a delicate technique:

$$|E_n^{GG}[u]| \leq \frac{\pi M(\rho^2 + 2 + \rho^{-2})}{2(\rho^{2n+2}-1)}. \quad (3.18)$$

We see that (3.17) is sharper. \square

For general $\alpha > -1$, the derivation of an explicit bound for

$$\Theta_n^\alpha := \max_{l \geq 0} \theta_{n,l}^\alpha, \quad \theta_{n,l}^\alpha := \gamma_n^{\alpha,\alpha} |\mu_{n,2l+2}^{\alpha,\alpha} - \mu_{n,2l}^{\alpha,\alpha}|, \quad n \geq 1, \quad (3.19)$$

seems nontrivial. We have only empirical results bases on computation. Some indications are listed as follows.

- (i) Observe from (3.16) that for fixed n , $\{\theta_{n,l}^{1/2}\}_{l \geq 0}$ are $(n+1)$ -periodic (see Figure 3.1 (a)), and the maximum is attained at $l = j(n+1)$, $j = 0, 1, \dots$. We compute ample samples of n, l and α , and find very similar “periodic” behaviors (see Figure 3.1 (b)-(c) for $\alpha = 0, 1$).
- (ii) Another interesting empirical observation is that for fixed α , the maximum value Θ_n^α converges to a constant value, and it decreases as α increases (see Figure 3.1 (d)). Note that for the Legendre case, $\Theta_n^0 \approx 4$.

Now, we turn to the second approach. The main result is summarized below.

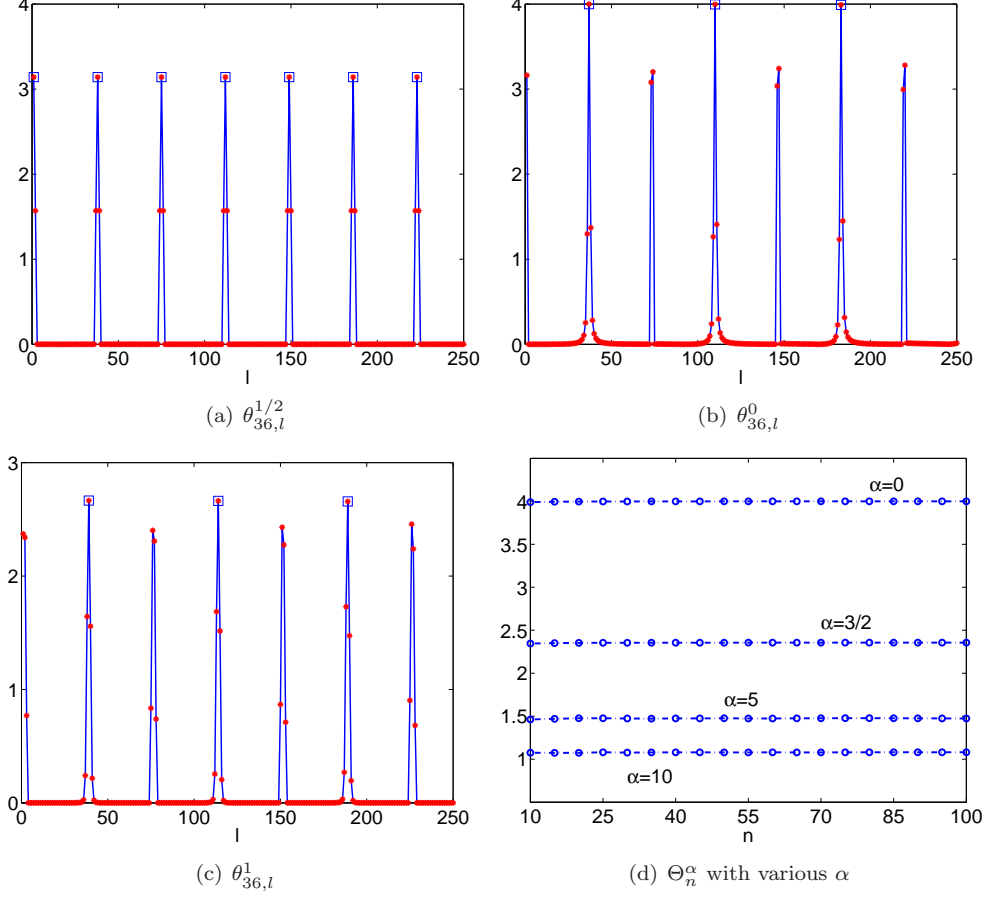


FIGURE 3.1. (a)-(c): Profiles of $\theta_{n,l}^{\alpha}$ with $n = 36, \alpha = 1/2, 0, 1$ and $0 \leq l \leq 250$. We mark by “ \square ” the location of the maximum value Θ_n^{α} is attached. (d): The maximum value Θ_n^{α} with $\alpha = 0, 3/2, 5, 10$ and $10 \leq n \leq 100$, where we compute $\{\theta_{n,l}^{\alpha}\}$ for l up to 1000.

Theorem 3.2. For any $u \in \mathcal{A}_{\rho}$ with $\rho > 1$, and for $\alpha > -1$ and $\alpha \neq -1/2$, we have

$$|E_n^{GG}[u]| \leq \frac{C_n M \sqrt{\pi}}{\rho^{2n}} \left(\frac{\sqrt{\pi}}{2^{2\alpha}} + \frac{\Gamma(\alpha+1)}{\sqrt{\Gamma(2\alpha+2)}} \frac{2\sqrt{2}}{\rho^2-1} \right) \begin{cases} (1+\rho^{-2})^{\alpha+1/2}, & \alpha > -1/2, \\ (1-\rho^{-2})^{\alpha+1/2}, & \alpha < -1/2, \end{cases} \quad (3.20)$$

and in particular, for the Legendre case,

$$|E_n^{LG}[u]| \leq \frac{C_n M \pi \sqrt{1+\rho^{-2}}}{\rho^{2n}} \left(1 + \frac{1}{2(\rho^2-1)} \right), \quad (3.21)$$

where the constant $C_n \approx 1$.

Proof. We carry out the proof by using the important relation, due to (3.1) and (B.1):

$$\begin{aligned} |E_n^{GG}[u]| &\leq \frac{\gamma_n^{\alpha,\alpha}}{\min_{z \in \mathcal{E}_\rho} |J_n^{\alpha,\alpha}(z)|} \left| \frac{1}{\pi i} \oint_{\mathcal{E}_\rho} Q_n^{\alpha,\alpha}(z) u(z) dz \right| \\ &\stackrel{(B.1)}{=} \frac{\gamma_n^{\alpha,\alpha} |\hat{u}_n^{\alpha,\alpha}|}{\min_{z \in \mathcal{E}_\rho} |J_n^{\alpha,\alpha}(z)|}. \end{aligned} \quad (3.22)$$

Since the numerator has been estimated in Theorem 2.1 (also see (2.36)), it suffices to deal with the denominator.

By [39, (4.6)], we have

$$\begin{aligned} |J_n^{\alpha,\alpha}(z)| &\geq C_n |A_n^\alpha| \frac{n^{\alpha-1/2} \rho^n}{|\Gamma(\alpha+1/2)|} \begin{cases} (1+\rho^{-2})^{-\alpha-1/2}, & \text{if } \alpha > -1/2, \\ (1-\rho^{-2})^{-\alpha-1/2}, & \text{if } \alpha < -1/2, \end{cases} \\ &\geq C_n \frac{2^{2\alpha} \rho^n}{\sqrt{\pi n}} \begin{cases} (1+\rho^{-2})^{-\alpha-1/2}, & \text{if } \alpha > -1/2, \\ (1-\rho^{-2})^{-\alpha-1/2}, & \text{if } \alpha < -1/2, \end{cases} \end{aligned} \quad (3.23)$$

where $C_n \approx 1$. Note that in the last step, we dealt with $|A_n^\alpha|$ as

$$|A_n^\alpha| \stackrel{(3.8)}{=} \frac{|\Gamma(2\alpha+1)|}{\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+2\alpha+1)} = \frac{|\Gamma(\alpha+1/2)|}{2^{-2\alpha} \sqrt{\pi}} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+2\alpha+1)} \geq C_n \frac{|\Gamma(\alpha+1/2)|}{2^{-2\alpha} \sqrt{\pi n}^\alpha},$$

where we used Lemma 2.1 and the property of Gamma function (see [1]):

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z} \sqrt{\pi} \Gamma(2z).$$

Therefore, a combination of (2.12), (2.36) and (3.22)-(3.23) leads to the desired result.

Using the refined estimate (2.50), yields (3.21). \square

4. NUMERICAL RESULTS AND COMPARISONS

In this section, we present various numerical results to show the tightness of the bounds derived in this paper, and to compare them with other existing ones mentioned in the previous part.

In the first example, we purposely choose the Chebyshev and Legendre expansions with known expansion coefficients:

$$u_1(x) = \frac{3}{5-4x} = T_0(x) + \sum_{n=1}^{\infty} \frac{T_n(x)}{2^{n-1}}, \quad u_2(x) = \frac{2}{\sqrt{5-4x}} = \sum_{n=0}^{\infty} \frac{L_n(x)}{2^n}, \quad (4.1)$$

which follow from generating functions of Chebyshev and Legendre polynomials (cf. [33]).

Note that the function u_1 has a simple pole at $z = 5/4$, so the semi-major axis (cf. (2.3)) should satisfy

$$1 < a = (\rho + \rho^{-1})/2 < 5/4 \Rightarrow 1 < \rho < 2.$$

One also verifies that

$$M = \max_{z \in \mathcal{E}_\rho} |u_1(z)| = \frac{3\rho}{(2\rho-1)(2-\rho)}.$$

Then the estimate (2.40) reduces to

$$\hat{u}_n^C = \frac{1}{2^{n-1}} \leq \frac{6}{(2\rho-1)(2-\rho)\rho^{n-1}} := B_n^C(\rho), \quad 1 < \rho < 2, \quad n \geq 1.$$

Similarly, for the Legendre expansion of u_2 , the result (2.50) becomes

$$\hat{u}_n^{0,0} = \frac{1}{2^n} \leq \frac{\sqrt{\pi n}}{\rho^n} \left(1 + \frac{n+2}{2n+3} \frac{1}{\rho^2-1} \right) \exp \left(\frac{8n-1}{12n(2n-1)} \right) \sqrt{\frac{4\rho}{(2\rho-1)(2-\rho)}} := B_n^L(\rho),$$

for $1 < \rho < 2$ and $n \geq 1$.

We take $\rho = 1.98$, and plot the exact coefficients \hat{u}_n^C and $\hat{u}_n^{0,0}$, and the bounds B_n^C and B_n^L in Figure 4.1 (a) and (b), respectively. Actually, the bound for the Chebyshev case (see (1.1)) can be considered as one benchmark for illustrating tightness of the upper bound. Indeed, the result for the Legendre case stated in Theorem 2.2 seems as sharp as that for the Chebyshev case.

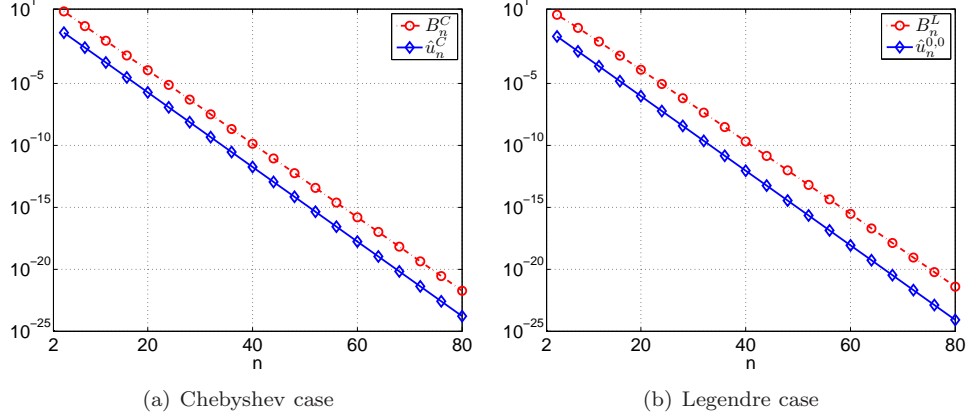


FIGURE 4.1. Expansion coefficients of u_1, u_2 in (4.1) against their error bounds.

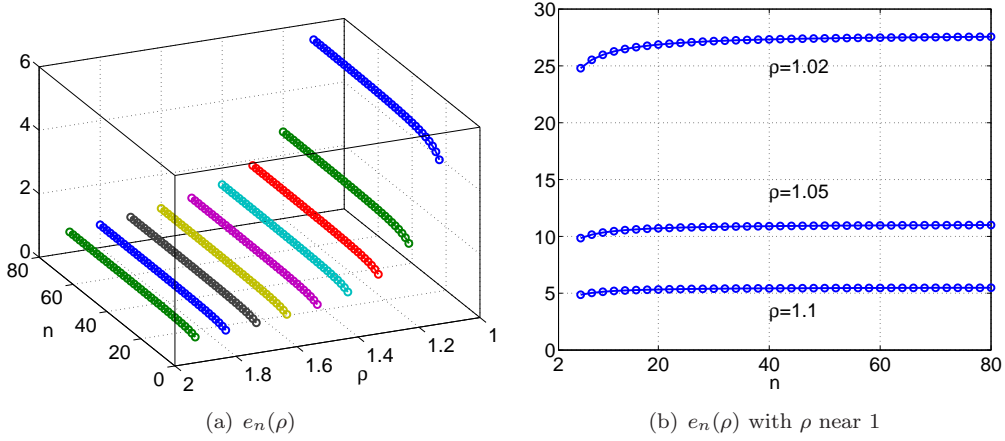


FIGURE 4.2. (a): Comparison of error bounds for Legendre expansions in (2.50) and (2.56). (b): Samples of $e_n(\rho)$ for ρ close to 1.

Next, we compare the bounds for the Legendre expansion coefficients in Theorem 2.2 and (2.56) (obtained by [38]). For clarity, we drop the common part $M\sqrt{n}/\rho^n$, and denote the remaining factors in the upper bounds by

$$b_n(\rho) \stackrel{(2.50)}{=} \sqrt{\pi} \left(1 + \frac{n+2}{2n+3} \frac{1}{\rho^2-1} \right) \exp \left(\frac{8n-1}{12n(2n-1)} \right), \quad \tilde{b}(\rho) \stackrel{(2.56)}{=} 2 \left(1 + \frac{1}{\rho^2-1} \right).$$

In Figure 4.2 (a), we plot the difference $e_n(\rho) := \tilde{b}(\rho) - b_n(\rho)$ for various ρ and $1 \leq n \leq 80$. We see that $e_n(\rho) > 0$ and the difference is of magnitude around 6, when ρ is close to 1. Moreover, for fixed ρ , the difference is roughly a constant for slightly large n . In Figure 4.2 (b), we plot some sample $e_n(\rho)$ for ρ close to 1, we see that our bound is much sharper.

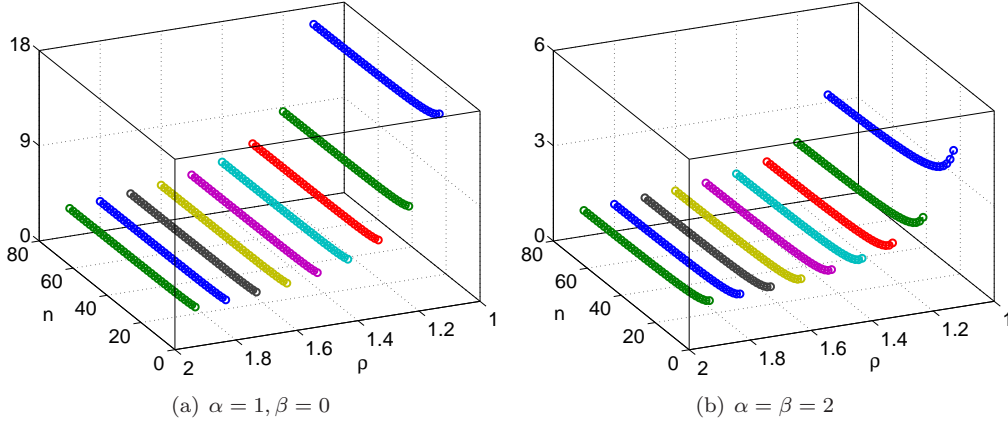


FIGURE 4.3. (a): Comparison of error bounds for Jacobi expansion with $\alpha = 1, \beta = 0$ in (1.2) and (2.58). (b): Comparison of error bounds for Gegenbauer expansion with $\alpha = \beta = 2$ in (1.2) and (2.27).

We next make a similar comparison of bounds for Jacobi and Gegenbauer expansions. For example, for $\alpha = 1$ and $\beta = 0$, we extract the factors in (1.2) and (2.58) by dropping $M\sqrt{n}/\rho^n$. We plot in Figure 4.3 (a) the difference of two remaining parts (i.e., that of (1.2) subtracts that of (2.58)). Once again, our bound is much tighter. Likewise, we depict in Figure 4.3 (b) the extracted bounds from (1.2) and (2.27) with $\alpha = \beta = 2$. The situation is mimic to the Legendre case, where the bounds obtained in this paper are sharper.

Finally, we turn to the comparison of error bounds for the Gegenbauer-Gauss quadrature remainder. For $\alpha = 1/2$, we extract the factors in (3.17) and (3.18) by dropping M/ρ^{2n} as before. We plot in Figure 4.4 (a) the difference of two remaining parts in (3.18) and in (3.17)). Once again, our bound is much tighter. Likewise, we depict in Figure 4.4 (b) the extracted bounds from (3.4) and (3.11) with $\alpha = 2$, and observe similar behaviors.

Concluding remarks

In this paper, we derived various new and sharp error bounds for Jacobi polynomial expansions and Gegenbauer-Gauss quadrature of analytic functions with analyticity characterized by the Bernstein ellipse. We adopted an argument that could recover the best known bounds, and attempted to make the dependence of the estimates on the parameters explicitly. Both analytic estimates and numerical comparisons with available ones demonstrated the sharpness of the error bounds.

APPENDIX A. JACOBI POLYNOMIALS

We collect some properties of Jacobi polynomials used in the paper. For $\alpha, \beta > -1$, the Jacobi polynomials (see e.g., [33]), denoted by $J_n^{\alpha, \beta}(x), x \in I := (-1, 1)$, are defined by the

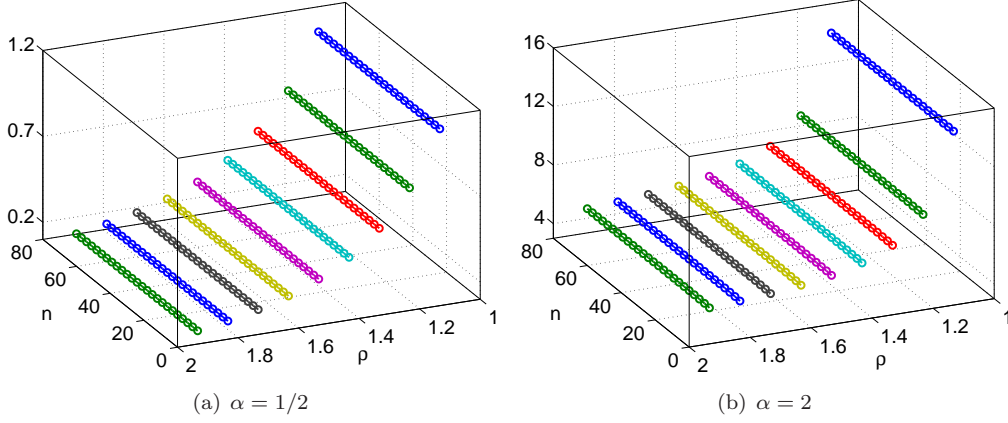


FIGURE 4.4. (a): Comparison of error bounds for the Gegenbauer-Gauss quadrature with $\alpha = 1/2$ in (3.17) and (3.18). (b): Comparison of error bounds for the Gegenbauer-Gauss quadrature with $\alpha = 2$ in (3.4) and (3.11).

Rodrigues' formula

$$(1-x)^\alpha(1+x)^\beta J_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1-x)^{\alpha+n}(1+x)^{\beta+n}], \quad n \geq 0. \quad (\text{A.1})$$

The Jacobi polynomials satisfy

$$(1-x)J_n^{\alpha+1,\beta}(x) = \frac{2}{2n+\alpha+\beta+2} ((n+\alpha+1)J_n^{\alpha,\beta}(x) - (n+1)J_{n+1}^{\alpha,\beta}(x)), \quad (\text{A.2a})$$

$$(1+x)J_n^{\alpha,\beta+1}(x) = \frac{2}{2n+\alpha+\beta+2} ((n+\beta+1)J_n^{\alpha,\beta}(x) + (n+1)J_{n+1}^{\alpha,\beta}(x)). \quad (\text{A.2b})$$

As a direct consequence of (A.2), we have that for any $k, l \in \mathbb{N} = \{0, 1, \dots\}$,

$$(1-x)^k(1+x)^l J_n^{\alpha+k,\beta+l}(x) = \sum_{i=n}^{n+k+l} d_i^{\alpha+k,\beta+l} J_i^{\alpha,\beta}(x), \quad (\text{A.3})$$

where $\{d_i^{\alpha+k,\beta+l}\}_{i=n}^{n+k+l}$ is a unique set of constants (with $d_n^{\alpha,\beta} = 1$), computed from (A.2) recursively. Here, we sketch the proof of (A.3). To this end, let $\{c_j\}$ be a set of generic constants. Using (A.2a) and (A.2b) repeatedly leads to

$$\begin{aligned} & (1-x)^k(1+x)^l J_n^{\alpha+k,\beta+l}(x) \\ &= (1-x)^{k-1}(1+x)^l (c_1 J_n^{\alpha+k-1,\beta+l}(x) + c_2 J_{n+1}^{\alpha+k-1,\beta+l}(x)) \\ &= \dots = (1+x)^l \sum_{m=n}^{n+k} c_m J_m^{\alpha,\beta+l}(x) = \dots = \sum_{m=n}^{n+k+l} c_m J_m^{\alpha,\beta}(x). \end{aligned}$$

This yields (A.3). We point out that for $\alpha = \beta = 0$, $\{(1-x)^k(1+x)^l J_n^{k,l}\}$ (up to a certain constant factor) are defined as generalized Jacobi polynomials in [22].

The following formula, derived from [2, Lemma 7.1.1] (also see [32, Theorem 3.21]), was used for the derivation of (2.21):

$$\begin{aligned}\hat{c}_j^n &:= \hat{c}_j^n(\alpha, \beta, a, b) = \frac{1}{\gamma_n^{\alpha, \beta}} \int_{-1}^1 J_{n+j}^{a, b}(x) J_n^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x) dx \\ &= \frac{\Gamma(n+j+a+1)}{\Gamma(n+j+a+b+1)} \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)} \\ &\quad \times \sum_{m=0}^j \frac{(-1)^m \Gamma(2n+j+m+a+b+1) \Gamma(n+m+\alpha+1)}{m!(j-m)! \Gamma(n+m+a+1) \Gamma(2n+m+\alpha+\beta+2)},\end{aligned}\tag{A.4}$$

for $a, b, \alpha, \beta > -1$ and $n, j \geq 0$.

Let $T_n(x) = \cos(n \arccos(x))$ be the Chebyshev polynomial of the first kind of degree n . Then the second-kind Chebyshev polynomial, denoted by $U_n(x)$, can be expressed by

$$U_n(x) = \frac{\sin((n+1) \arccos(x))}{\sqrt{1-x^2}} = \frac{T'_{n+1}(x)}{n+1} = \sqrt{\frac{\pi}{2}} \frac{J_n^{1/2, 1/2}(x)}{\sqrt{\gamma_n^{1/2, 1/2}}}.\tag{A.5}$$

The Chebyshev polynomials enjoy the following important properties:

$$J_n^{-1/2, -1/2}(x) = J_n^{-1/2, -1/2}(1) T_n(x) = \frac{\Gamma(n+1/2)}{\sqrt{\pi} n!} T_n(x),\tag{A.6a}$$

$$T'_n(x) = 2n \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{n-1} \frac{1}{c_k} T_k(x),\tag{A.6b}$$

where $c_0 = 2$ and $c_k = 1$ for $k \geq 1$.

APPENDIX B. PROOF OF LEMMA 2.2

We first show that

$$\hat{u}_n^{\alpha, \beta} = \frac{1}{\pi i} \oint_{\mathcal{E}_\rho} Q_n^{\alpha, \beta}(z) u(z) dz,\tag{B.1}$$

where

$$Q_n^{\alpha, \beta}(z) := \frac{1}{2\gamma_n^{\alpha, \beta}} \int_{-1}^1 \frac{J_n^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x)}{z-x} dx,\tag{B.2}$$

and $\gamma_n^{\alpha, \beta}$ is given by (2.8). Recall the Cauchy's integral formula:

$$\frac{d^n}{dx^n} u(x) = \frac{n!}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{u(z)}{(z-x)^{n+1}} dz.\tag{B.3}$$

Using the Rodrigues' formula (A.1) and integration by parts leads to

$$\begin{aligned}\hat{u}_n^{\alpha, \beta} &\stackrel{(1.2)}{=} \frac{1}{\gamma_n^{\alpha, \beta}} \int_{-1}^1 u(x) J_n^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x) dx \\ &\stackrel{(A.1)}{=} \frac{1}{\gamma_n^{\alpha, \beta}} \frac{(-1)^n}{2^n n!} \int_{-1}^1 \omega^{\alpha+n, \beta+n}(x) \frac{d^n}{dx^n} u(x) dx \\ &\stackrel{(B.3)}{=} \frac{1}{\gamma_n^{\alpha, \beta}} \frac{1}{2^n n!} \int_{-1}^1 \left(\frac{n!}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{u(z)}{(z-x)^{n+1}} dz \right) \omega^{\alpha+n, \beta+n}(x) dx \\ &= \frac{1}{2^n \gamma_n^{\alpha, \beta}} \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \left(\int_{-1}^1 \frac{\omega^{\alpha+n, \beta+n}(x)}{(z-x)^{n+1}} dx \right) u(z) dz.\end{aligned}\tag{B.4}$$

We find from integration by parts that

$$\int_{-1}^1 \frac{\omega^{\alpha+n, \beta+n}(x)}{(z-x)^{n+1}} dx = \frac{(-1)^n}{n!} \int_{-1}^1 \frac{1}{z-x} \frac{d^n}{dx^n} \omega^{\alpha+n, \beta+n}(x) dx. \quad (\text{B.5})$$

Inserting (B.5) into (B.4), we derive from the Rodrigues' formula (A.1) that

$$\hat{u}_n^{\alpha, \beta} = \frac{1}{2\pi i} \frac{1}{\gamma_n^{\alpha, \beta}} \oint_{\mathcal{E}_\rho} \left(\int_{-1}^1 \frac{\omega^{\alpha, \beta}(x) J_n^{\alpha, \beta}(x)}{z-x} dx \right) u(z) dz = \frac{1}{\pi i} \oint_{\mathcal{E}_\rho} Q_n^{\alpha, \beta}(z) u(z) dz,$$

where $Q_n^{\alpha, \beta}(z)$ is given in (B.2).

Since $z = (w+w^{-1})/2$, we have from the generating function of the Chebyshev polynomial of the second-kind (cf. [1]) that

$$\frac{1}{z-x} = \frac{2}{w} \frac{1}{w^{-2} - 2xw^{-1} + 1} = \frac{2}{w} \sum_{k=0}^{\infty} \frac{U_k(x)}{w^k}. \quad (\text{B.6})$$

Inserting it into (B.2), we find from the orthogonality of the Jacobi polynomials (cf. (2.7)) that

$$\begin{aligned} Q_n^{\alpha, \beta}(z) &= \frac{1}{\gamma_n^{\alpha, \beta}} \sum_{k=0}^{\infty} \frac{1}{w^{k+1}} \int_{-1}^1 U_k(x) J_n^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x) dx \\ &= \frac{1}{\gamma_n^{\alpha, \beta}} \sum_{k=n}^{\infty} \frac{1}{w^{k+1}} \int_{-1}^1 U_k(x) J_n^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x) dx \\ &= \frac{1}{\gamma_n^{\alpha, \beta}} \sum_{j=0}^{\infty} \frac{1}{w^{n+j+1}} \int_{-1}^1 U_{n+j}(x) J_n^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x) dx = \sum_{j=0}^{\infty} \frac{\sigma_{n,j}^{\alpha, \beta}}{w^{n+j+1}}, \end{aligned} \quad (\text{B.7})$$

where we defined

$$\sigma_{n,j}^{\alpha, \beta} = \frac{1}{\gamma_n^{\alpha, \beta}} \int_{-1}^1 U_{n+j}(x) J_n^{\alpha, \beta}(x) \omega^{\alpha, \beta}(x) dx,$$

Substituting the last identity of (B.7) into (B.1) leads to the desired formula (2.15).

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